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Jordan and Jordan Higher All-derivable Points of Some Algebras

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Abstract

In this paper, we characterize Jordan derivable mappings in terms of Peirce decomposition and determine Jordan all-derivable points for some general bimodules. Then we generalize the results to the case of Jordan higher derivable mappings. An immediate application of our main results shows that for a nest $\mathcal{N}$ on a Banach X with the associated nest algebra $alg\mathcal{N}$, if there exists a non-trivial element in $\mathcal{N}$ which is complemented in X, then every $C \in alg\mathcal{N}$ is a Jordan all-derivable point of $L(alg\mathcal{N}, B(X))$ and a Jordan higher all-derivable point of $L(alg\mathcal{N})$.

Keywords : Jordan all-derivable point, Jordan derivation, Jordan higher all-derivable point, Jordan higher derivation; nest algebra

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1 Introduction

Let $\mathcal{A}$ be a unital algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. We denote $C(\mathcal{A},\mathcal{M}) = \{M \in \mathcal{M} : AM = MA \text{ for every } A \in \mathcal{A}\}$ and $L(\mathcal{A},\mathcal{M})$ the set of all linear mappings from $\mathcal{A}$ to $\mathcal{M}$. When $\mathcal{M} = \mathcal{A}$, we relabel $L(\mathcal{A},\mathcal{M})$ as $L(\mathcal{A})$. Let $\delta \in L(\mathcal{A},\mathcal{M})$. δ is called a *derivation* if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$; it is a *Jordan derivation* if $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ for all $A, B \in \mathcal{A}$; it is a *generalized derivation* if there exists an $M_\delta \in C(\mathcal{A},\mathcal{M})$ such that $\delta(AB) = \delta(A)B + A\delta(B) - M_\delta AB$ for all $A, B \in \mathcal{A}$. For any fixed $M \in \mathcal{M}$, each mapping of the form $\delta_M(A) = MA - AM$ for every $A \in \mathcal{A}$ is called an *inner derivation*. Clearly each inner derivation is a derivation and each derivation in a Jordan derivation. But the converse is not true in general. The questions of characterizing derivations and Jordan derivations have received considerable attention from several authors, who revealed

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the relations among derivations, Jordan derivations as well as inner derivations (see for example [3, 4, 10, 14, 23, 28, 32], and the references therein).

In general there are two directions in the study of the local actions of derivations of operator algebras. One is the well known local derivation problem (see for example [7, 9, 15, 35, 40]). The other is to study conditions under which derivations of operator algebras can be completely determined by the action on some subsets of operators (see for example [3, 5, 17, 30, 34, 39]). A mapping $\delta \in L(\mathcal{A}, \mathcal{M})$ is called a *Jordan derivable mapping at* $C \in \mathcal{A}$ if $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ for all $A, B \in \mathcal{A}$ with $AB = C$. It is obvious that a linear mapping is a Jordan derivation if and only if it is Jordan derivable at all points. It is natural and interesting to ask the question whether or not a linear mapping is a Jordan derivation if it is Jordan derivable only at one given point. If such a point exists, we call this point a Jordan all-derivable point. To be more precise, an element $C \in \mathcal{A}$ is called a *Jordan all-derivable point* of $L(\mathcal{A}, \mathcal{M})$ if every Jordan derivable mapping at C is a Jordan derivation. It is quite surprising that there do exist Jordan all-derivable points for some algebras. An and Hou [2] show that under some mild conditions on unital prime ring or triangular ring $\mathcal{A}$, I is a Jordan all-derivable point of $L(\mathcal{A})$. Jiao and Hou [13] study Jordan derivable mappings at zero point on nest algebras. Zhao and Zhu [33] prove that 0 and I are Jordan all-derivable points of the triangular algebra. In [16], the authors study some derivable mappings in the generalized matrix algebra $\mathcal{A}$, and show that 0, P and I are Jordan all-derivable points, where P is the standard non-trivial idempotent. In [33], Zhao and Zhu prove that every element in the algebra of all $n \times n$ upper triangular matrices over the complex field $\mathbb{C}$ is a Jordan all-derivable point. In Section 2, we give some general characterizations of Jordan derivable mappings, which will be used to determine Jordan all-derivable points for some general bimodules.

Let $\mathcal{A}$ be a unital algebra and $\mathbb{N}$ be the set of non-negative integers. A sequence of mappings $\{d_i\}_{i\in\mathbb{N}} \in L(\mathcal{A})$ with $d_0 = I_{\mathcal{A}}$ is called a *higher derivation* if $d_n(AB) = \sum_{i+j=n} d_i(A)d_j(B)$ for all $A, B \in \mathcal{A}$; it is called a *Jordan higher derivation* if $d_n(AB + BA) = \sum_{i+j=n}(d_i(A)d_j(B) + d_i(B)d_j(A))$ for all $A, B \in \mathcal{A}$. With the development of derivations, the study of higher and Jordan higher derivations has attracted much attention as an active subject of research in operator algebras, and the local action problem ranks among in the list. A sequence of mappings $\{d_i\}_{i\in\mathbb{N}} \in L(\mathcal{A})$ with $d_0 = I_{\mathcal{A}}$ is called *Jordan higher derivable at* $C \in \mathcal{A}$ if $d_n(AB + BA) = \sum_{i+j=n}(d_i(A)d_j(B) + d_i(B)d_j(A))$ for all $A, B \in \mathcal{A}$ with $AB = C$. An element $C \in \mathcal{A}$ is called a *Jordan higher all-derivable point* if every sequence of Jordan higher derivable mappings at C is a Jordan higher derivation. In Section 3, we generalize the results in Section 2 to the case of Jordan higher derivable mappings. Meanwhile, we find the connection between Jordan all-derivable points (all-derivable points, S-Jordan all derivable points, respectively) and Jordan higher all-derivable points (higher all-derivable points, S-Jordan higher all-derivable points, respectively). We also discuss the automatic continuity property of (Jordan) higher derivations.

Let X be a complex Banach space and $B(X)$ be the set of all bounded linear operators on X. For any non-empty subset $L \subseteq X$, $L^{\perp}$ denotes its annihilator, that is, $L^{\perp} = \{f \in X^* : f(x) = 0 \text{ for all } x \in L\}$. By a *subspace lattice* on X, we mean a collection $\mathcal{L}$ of closed subspaces of X with (0) and X in $\mathcal{L}$ such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\cap M_r$ and

$\vee M_r$ belong to $\mathcal{L}$. For a subspace lattice $\mathcal{L}$ of X, let alg$\mathcal{L}$ denote the algebra of all operators in $B(X)$ that leave members of $\mathcal{L}$ invariant. A totally ordered subspace lattice is called a *nest*. If $\mathcal{L}$ is a nest, then alg$\mathcal{L}$ is called a *nest algebra*, see [8] for more on nest algebras. When X is a separable Hilbert space over the complex field $\mathbb{C}$, we change it to H. In a Hilbert space, we disregard the distinction between a closed subspace and the orthogonal projection onto it. An immediate but noteworthy application of our main result shows that for a nest $\mathcal{N}$ on a Banach X with the associated nest algebra $alg\mathcal{N}$, if there exists a non-trivial element in $\mathcal{N}$ which is complemented in X, then every $C \in alg\mathcal{N}$ is a Jordan all-derivable point of $L(alg\mathcal{N}, B(X))$ and a Jordan higher all-derivable point of $L(alg\mathcal{N})$.

2 Jordan derivable mappings

We start with Peirce decomposition of algebras and its bimodules.

Let $\mathcal{A}$ be a unital algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. For any idempotent $E_1 \in \mathcal{A}$, let $E_2 = I - E_1$. For $i, j \in \{1, 2\}$, define $\mathcal{A}_{ij} = E_i\mathcal{A}E_j$, which gives the Peirce decomposition of $\mathcal{A} : A = A_{11} + A_{12} + A_{21} + A_{22}$. Similarly, we define $\mathcal{M}_{ij} = E_i\mathcal{M}E_j$. We say $\mathcal{A}_{ij}$ is *left faithful* with respect to $\mathcal{M}$ if for any $M \in \mathcal{M}$, the condition $M\mathcal{A}_{ij} = \{0\}$ implies $ME_i = 0$ and $\mathcal{A}_{ij}$ is *right faithful* with respect to $\mathcal{M}$ if the condition $\mathcal{A}_{ij}M = \{0\}$ implies $E_jM = 0$. We say $\mathcal{A}_{ij}$ is *faithful* with respect to $\mathcal{M}$ if it is both left faithful and right faithful. In this paper, we will always use the notations $P = E_1$ and $Q = E_2 = I - E_1$ for convenience.

In this section, we will assume $\mathcal{A}$ is a unital algebra over a field $\mathbb{F}$ of characteristic not equal to 2 and $|\mathbb{F}| \geq 4$, $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule and $\mathcal{A}$ has a non-trivial idempotent $P = E_1 \in \mathcal{A}$ such that the corresponding Peirce decomposition has the following property: Every element of A_{11} is a linear combination of invertible elements of A_{11} and every element of A_{22} is a linear combination of invertible elements of A_{22}. Algebras satisfying these assumptions include all finite-dimensional unital algebras over an algebraically closed field and all unital Banach algebras.

For any $A, B \in \mathcal{A}$, define $A \circ B = AB + BA$; similarly, for any $A \in \mathcal{A}$ and $M \in \mathcal{M}$, define $A \circ M = AM + MA$. For $A, B, D, E \in \mathcal{A}$, we say any $\delta \in L(\mathcal{A}, \mathcal{M})$ *differentiates* $A \circ B$ if $\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$; we say δ *differentiates* $A \circ B + C \circ D$ if $\delta(A \circ B + C \circ D) = \delta(A) \circ B + A \circ \delta(B) + \delta(D) \circ E + D \circ \delta(E)$. We see that a mapping $\delta \in L(\mathcal{A}, \mathcal{M})$ is a Jordan derivation if and only if δ differentiates $A \circ B$ for all $A, B \in \mathcal{A}$, and δ is Jordan derivable at $C \in \mathcal{A}$ if and only if δ differentiates $A \circ B$ for all $A, B \in \mathcal{A}$ with $AB = C$.

The following proposition is elementary, we omit the proof.

Proposition 2.1. *Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$ with $|\mathbb{F}| > n$. For any fixed $v_i \in \mathcal{V}, i = 0, 1, \cdots, n$, define $p(t) = \sum_{i=0}^{n} v_i t^i$ for $t \in \mathbb{F}$. If $p(t) = 0$ has at least $n+1$ distinct solutions in $\mathbb{F}$, then $v_i = 0,\ i = 0, 1, \cdots, n$.*

A simple application of Proposition 2.1 yields the following proposition, which will be used repeatedly in this paper.

Proposition 2.2. *Suppose $A, B, D, E, K, L \in \mathcal{A}$ and $\delta \in L(\mathcal{A}, \mathcal{M})$.*

(a) If δ differentiates $(tA+B)\circ(tD+E)$ for at least three $t\in\mathbb{F}$, then δ differentiates $A\circ D$, $B\circ E$, and $A\circ E+B\circ D$; in particular, if $A=0$ then δ differentiates $B\circ D$.

(b) If δ differentiates $A\circ(tD+E)+B\circ(tK+L)$ for at least two $t\in\mathbb{F}$, then δ differentiates $A\circ D+B\circ K$ and $A\circ E+B\circ L$.

Now we characterize Jordan-derivable mappings in terms of Peirce decomposition as follows.

Theorem 2.3. *For any $C\in\mathcal{A}$ such that $C_{21}=0$, if $\Delta\in L(\mathcal{A},\mathcal{M})$ is Jordan-derivable at C, then there exists a $\delta\in L(\mathcal{A},\mathcal{M})$ such that $\Delta-\delta$ is an inner derivation and the following hold:*

(a) $\delta(P)A_{12}=A_{12}\delta(Q)$ for any $A_{12}\in\mathcal{A}_{12}$.

(b) $A_{12}\delta(A_{12})=\delta(A_{12})A_{12}=0$ for any $A_{12}\in\mathcal{A}_{12}$.

(c) $\delta(\mathcal{A}_{11})\subset\mathcal{M}_{11}$, $\delta(\mathcal{A}_{22})\subset\mathcal{M}_{22}$.

If $\mathcal{A}_{12}$ is left faithful, then

(d) $\delta(P)\in C(\mathcal{A}_{11},\mathcal{M})$.

(e) $\delta|_{\mathcal{A}_{11}}$ is a generalized derivation from $\mathcal{A}_{11}$ to $\mathcal{M}_{11}$.

If $\mathcal{A}_{12}$ is right faithful, then

(f) $\delta(Q)\in C(\mathcal{A}_{22},\mathcal{M})$.

(g) $\delta|_{\mathcal{A}_{22}}$ is a generalized derivation from $\mathcal{A}_{22}$ to $\mathcal{M}_{22}$.

Proof. Let $M=P\Delta(Q)Q-Q\Delta(Q)P$ and define $\delta(A)=\Delta(A)-(MA-AM)$ for every $A\in\mathcal{A}$. Then δ is Jordan-derivable at any $G\in\mathcal{A}$ if and only if Δ is Jordan-derivable at G; moreover $\delta(Q)\in\mathcal{M}_{11}+\mathcal{M}_{22}$ by direct computation. Write $C=C_{11}+C_{12}+C_{22}$. Fix any $A_{11}\in\mathcal{A}_{11}$ that is invertible in $\mathcal{A}_{11}$ with $A_{11}^{-1}\in\mathcal{A}_{11}$ and $Z_{22},W_{22}\in\mathcal{A}_{22}$ such that $Z_{22}W_{22}=C_{22}$. Note that we can take any W_{22} that is invertible in $\mathcal{A}_{22}$ with $W_{22}^{-1}\in\mathcal{A}_{22}$ and $Z_{22}=C_{22}W_{22}^{-1}$ to satisfy $Z_{22}W_{22}=C_{22}$. For any $0\neq t\in\mathbb{F}$, $s\in\mathbb{F}$, and $A_{12}\in\mathcal{A}_{12}$, a routine computation yields

$$[A_{11}+t(sA_{11}A_{12}+Z_{22})][(A_{11}^{-1}C-sA_{12}W_{22})+t^{-1}W_{22}]=C.$$

Since δ is Jordan derivable at C, δ differentiates

$$[A_{11}+t(sA_{11}A_{12}+Z_{22})]\circ[(A_{11}^{-1}C-sA_{12}W_{22})+t^{-1}W_{22}].$$

Thus δ differentiates $[A_{11}+t(sA_{11}A_{12}+Z_{22})]\circ[t(A_{11}^{-1}C-sA_{12}W_{22})+W_{22}]$. By Proposition 2.2($a$), we get ($i$) δ differentiates $A_{11}\circ W_{22}$, (ii) δ differentiates $(sA_{11}A_{12}+Z_{22})\circ(A_{11}^{-1}C-sA_{12}W_{22})$, and ($iii$) δ differentiates $A_{11}\circ(A_{11}^{-1}C-sA_{12}W_{22})+(sA_{11}A_{12}+Z_{22})\circ W_{22}$.

By (i), we get

$$\delta(A_{11})\circ W_{22}+A_{11}\circ\delta(W_{22})=\delta(A_{11}\circ W_{22})=0 \tag{2.1}$$

By (ii) and Proposition 2.2(a), we have δ differentiates $(A_{11}A_{12})\circ(A_{12}W_{22})$, i.e.

$$\delta(A_{11}A_{12})\circ(A_{12}W_{22})+(A_{11}A_{12})\circ\delta(A_{12}W_{22})=\delta[(A_{11}A_{12})\circ(A_{12}W_{22})]=0 \tag{2.2}$$

By (iii) and Proposition 2.2(b), we have

$$\begin{aligned}0=\delta[A_{11}\circ(-A_{12}W_{22})+(A_{11}A_{12})\circ W_{22}]&=\delta(A_{11})\circ(-A_{12}W_{22})+A_{11}\circ\delta(-A_{12}W_{22})\\&\quad+\delta(A_{11}A_{12})\circ W_{22}+(A_{11}A_{12})\circ\delta(W_{22})\end{aligned}$$

Thus

$$\delta(A_{11}A_{12})\circ W_{22}+(A_{11}A_{12})\circ\delta(W_{22})-\delta(A_{11})\circ(A_{12}W_{22})-A_{11}\circ\delta(A_{12}W_{22})=0 \tag{2.3}$$

Since $\delta(Q)\in\mathcal{M}_{11}+\mathcal{M}_{22}$, $A_{11}\circ\delta(Q)\in\mathcal{M}_{11}$. Setting $W_{22}=Q$ in Eq. (2.1) gives

$$\delta(A_{11})\circ Q+A_{11}\circ\delta(Q)=0$$

Thus $A_{11}\circ\delta(Q)=\delta(A_{11})\circ Q=0$. It follows $\delta(A_{11})Q=Q\delta(A_{11})=0$. Hence $\delta(A_{11})\in\mathcal{M}_{11}$, and $\delta(A_{11})\circ W_{22}=0$. By Eq. (2.1) again, we get $A_{11}\circ\delta(W_{22})=0$; in particular, $P\circ\delta(W_{22})=0$. It follows that $\delta(W_{22})\in\mathcal{M}_{22}$, which proves (c).

Taking $A_{11}=P$ in Eq. (2.3) yields

$$\delta(A_{12})\circ W_{22}+A_{12}\circ\delta(W_{22})-\delta(P)\circ(A_{12}W_{22})-P\circ\delta(A_{12}W_{22})=0 \tag{2.4}$$

Multiplying P from both sides of Eq. (2.4) gives $P\delta(A_{12}W_{22})P=0$. In particular,

$$P\delta(A_{12})P=0 \tag{2.5}$$

Multiplying P from the left of Eq. (2.4) and applying Eq. (2.5), we get

$$P\delta(A_{12})W_{22}+A_{12}\delta(W_{22})-\delta(P)A_{12}W_{22}-P\delta(A_{12}W_{22})=0 \tag{2.6}$$

Setting $W_{22}=Q$ in Eq. (2.6) and combining with Eq. (2.5) leads to

$$A_{12}\delta(Q)=\delta(P)A_{12} \tag{2.7}$$

This proves (a).

Taking $A_{11}=P$ and $W_{22}=Q$ in Eq. (2.2), we get $A_{12}\circ\delta(A_{12})=0$, i.e.

$$A_{12}\delta(A_{12})+\delta(A_{12})A_{12}=0 \tag{2.8}$$

Multiplying P from the left of Eq. (2.8) and applying Eq. (2.5), yields $A_{12}\delta(A_{12})=0$; which gives $\delta(A_{12})A_{12}=0$, when applied to Eq. (2.8). This proves (b).

Taking $W_{22}=Q$ in Eq. (2.3) yields

$$\delta(A_{11}A_{12})\circ Q+(A_{11}A_{12})\circ\delta(Q)-\delta(A_{11})\circ A_{12}-A_{11}\circ\delta(A_{12})=0 \tag{2.9}$$

Multiplying Q from both sides of Eq. (2.9) gives $Q\delta(A_{11}A_{12})Q=0$. In particular,

$$Q\delta(A_{12})Q=0 \tag{2.10}$$

Multiplying Q from the right of Eq. (2.9) and applying Eq. (2.10) gives

$$\delta(A_{11}A_{12})Q+A_{11}A_{12}\delta(Q)-\delta(A_{11})A_{12}-A_{11}\delta(A_{12})Q=0$$

Combining this with Eq. (2.5) yields

$$\delta(A_{11}A_{12})Q=\delta(A_{11})A_{12}+A_{11}\delta(A_{12})-A_{11}A_{12}\delta(Q) \tag{2.11}$$

Replacing A_{11} with $A_{11}U_{11}$ in Eq. (2.11) gives

$$\delta(A_{11}U_{11}A_{12})Q = \delta(A_{11}U_{11})A_{12} + A_{11}U_{11}\delta(A_{12}) - A_{11}U_{11}A_{12}\delta(Q) \tag{2.12}$$

On the other hand, applying Eq. (2.11) twice gives

$$\begin{aligned}\delta(A_{11}U_{11}A_{12})Q &= A_{11}\delta(U_{11}A_{12}) + \delta(A_{11})U_{11}A_{12} - A_{11}U_{11}A_{12}\delta(Q)\\ &= A_{11}\delta(U_{11}A_{12})Q + \delta(A_{11})U_{11}A_{12} - A_{11}U_{11}A_{12}\delta(Q)\\ &= A_{11}[\delta(U_{11})A_{12} + U_{11}\delta(A_{12}) - U_{11}A_{12}\delta(Q)]\\ &\quad + \delta(A_{11})U_{11}A_{12} - A_{11}U_{11}A_{12}\delta(Q)\end{aligned} \tag{2.13}$$

By Eqs. (2.12), (2.13), and (2.7), we have

$$\delta(A_{11}U_{11})A_{12} = [\delta(A_{11})U_{11} + A_{11}\delta(U_{11}) - \delta(P)A_{11}U_{11}]A_{12}$$

If $\mathcal{A}_{12}$ is left faithful,

$$\delta(A_{11}U_{11}) = \delta(A_{11})U_{11} + A_{11}\delta(U_{11}) - \delta(P)A_{11}U_{11} \tag{2.14}$$

Taking $U_{11} = P$ in Eq. (2.14) gives $A_{11}\delta(P) = \delta(P)A_{11}$, that is, $\delta(P) \in C(\mathcal{A}_{11}, \mathcal{M})$. This proves (d) and now (e) follows directly from Eq. (2.14).

Since $\delta(P)A_{12} = A_{12}\delta(Q)$ for any $A_{12} \in \mathcal{A}_{12}$, we have $A_{12}\delta(Q)A_{22} = \delta(P)A_{12}A_{22} = A_{12}A_{22}\delta(Q)$, then faithfulness of $\mathcal{A}_{12}$ leads to $\delta(Q)A_{22} = A_{22}\delta(Q)$, that is, $\delta(Q) \in C(\mathcal{A}_{22}, \mathcal{M})$. This proves (f).

By Eqs. (2.6) and (2.10),

$$P\delta(A_{12}W_{22}) = \delta(A_{12})W_{22} + A_{12}\delta(W_{22}) - \delta(P)A_{12}W_{22} \tag{2.15}$$

Replacing W_{22} with $V_{22}W_{22}$ in Eq. (2.15) gives

$$P\delta(A_{12}V_{22}W_{22}) = \delta(A_{12})V_{22}W_{22} + A_{12}\delta(V_{22}W_{22}) - \delta(P)A_{12}V_{22}W_{22} \tag{2.16}$$

On the other hand, applying Eq. (2.15) twice gives

$$\begin{aligned}P\delta(A_{12}V_{22}W_{22}) &= \delta(A_{12}V_{22})W_{22} + A_{12}V_{22}\delta(W_{22}) - \delta(P)A_{12}V_{22}W_{22}\\ &= P\delta(A_{12}V_{22})W_{22} + A_{12}V_{22}\delta(W_{22}) - \delta(P)A_{12}V_{22}W_{22}\\ &= [\delta(A_{12})V_{22} + A_{12}\delta(V_{22}) - \delta(P)A_{12}V_{22}]W_{22}\\ &\quad + A_{12}V_{22}\delta(W_{22}) - \delta(P)A_{12}V_{22}W_{22}\end{aligned} \tag{2.17}$$

By Eqs. (2.16), (2.17), and (2.7),

$$A_{12}\delta(V_{22}W_{22}) = A_{12}[\delta(V_{22})W_{22} + V_{22}\delta(W_{22}) - \delta(Q)V_{22}W_{22}]$$

Since $\mathcal{A}_{12}$ is left faithful,

$$\delta(V_{22}W_{22}) = \delta(V_{22})W_{22} + V_{22}\delta(W_{22}) - \delta(Q)V_{22}W_{22} \tag{2.18}$$

This proves (g). $\square$

Suppose $\mathcal{B}$ is an algebra containing $\mathcal{A}$ and shares the same identity with $\mathcal{A}$, then $\mathcal{B}$ is an $\mathcal{A}$-bimodule with respect to the multiplication and addition of $\mathcal{B}$. Let $\mathcal{T}_{\mathcal{A}} = \{A \in \mathcal{A} : A_{21} = 0\}$. The following proposition is contained in [29, Theorem 3.3], we include a proof here for completeness.

Proposition 2.4. *Suppose $\mathcal{A}_{12}$ is faithful to $\mathcal{B}$, $C(\mathcal{T}_{\mathcal{A}}, \mathcal{B}) = \mathbb{F}I$, and $B \in \mathcal{B}$. If $T_{12}BT_{12} = 0$ for every $T_{12} \in \mathcal{A}_{12}$, then $QBP = 0$.*

Proof. Suppose $T_{12}BT_{12} = 0$ for every $T_{12} \in \mathcal{A}_{12}$. For any non-zero $A_{12}, T_{12} \in \mathcal{A}_{12}$, we have $T_{12}BT_{12} = 0$, $A_{12}BA_{12} = 0$ and $(A_{12} + T_{12})B(A_{12} + T_{12}) = 0$. It follows that

$$A_{12}BT_{12} + T_{12}BA_{12} = 0. \tag{2.19}$$

For any $A_{11} \in \mathcal{A}_{11}$, replacing A_{12} in Eq. (2.19) with $A_{11}A_{12}$ gives

$$A_{11}A_{12}BT_{12} + T_{12}BA_{11}A_{12} = 0. \tag{2.20}$$

Multiplying A_{11} from the left of Eq. (2.19) gives

$$A_{11}A_{12}BT_{12} + A_{11}T_{12}BA_{12} = 0. \tag{2.21}$$

By Eq. (2.20) and Eq. (2.21), we have

$$T_{12}BA_{11}A_{12} = A_{11}T_{12}BA_{12}.$$

Since A_{12} is arbitrary and $\mathcal{A}_{12}$ is faithful, we have

$$T_{12}BA_{11} = A_{11}T_{12}BP. \tag{2.22}$$

Similarly, we have

$$A_{22}QBT_{12} = QBT_{12}A_{22}. \tag{2.23}$$

Let $\widetilde{B} = T_{12}BP - QBT_{12}$. It follows from Eqs. (2.19), (2.22) and (2.23) that $\widetilde{B}$ commutes with A_{12}, A_{11} and A_{22}, that is, $\widetilde{B} \in C(\mathcal{T}_{\mathcal{A}}, \mathcal{B})$. Hence there exists a $k \in \mathbb{F}$ such that $\widetilde{B} = kI$. It follows $T_{12}BP = kP$. Now $T_{12}BT_{12} = 0$ leads to $kT_{12} = 0$. Hence $k = 0$ and $T_{12}BP = 0$. Since T_{12} is arbitrary and $\mathcal{A}_{12}$ is faithful, we have $QBP = 0$. $\square$

Theorem 2.5. *Suppose $\mathcal{A}_{12}$ is faithful to $\mathcal{B}$ and $C(\mathcal{T}_{\mathcal{A}}, \mathcal{B}) = \mathbb{F}I$. If $\delta \in L(\mathcal{A}, \mathcal{B})$ is Jordan derivable at $C \in \mathcal{T}_{\mathcal{A}}$ then $\delta|_{\mathcal{T}_{\mathcal{A}}}$ is a derivation from $\mathcal{T}_{\mathcal{A}}$ to $\mathcal{B}$.*

Proof. Substracting an inner derivation from δ if necessary, we can assume δ satisfies the properties of Theorem 2.3. Thus, for any A_{12} and T_{12} in $\mathcal{A}_{12}$, we have $\delta(A_{12})A_{12} = 0$, $\delta(T_{12})T_{12} = 0$ and $\delta(A_{12} + T_{12})(A_{12} + T_{12}) = 0$. It follows that $\delta(A_{12})T_{12} + \delta(T_{12})A_{12} = 0$. Multiplying T_{12} from the left we obtain $T_{12}\delta(A_{12})T_{12} = 0$. Since T_{12} is arbitrary, $Q\delta(A_{12})P = 0$, by Proposition 2.4. This, together with Eqs. (2.5) and (2.10), yields $\delta(A_{12}) \in \mathcal{B}_{12}$.

For any $A_{11} \in \mathcal{A}_{11}$ and $A_{22} \in \mathcal{A}_{22}$, by Theorem 2.3 $\delta(A_{11}) \in \mathcal{B}_{11}$ and $\delta(A_{22}) \in \mathcal{B}_{22}$.

Since $\delta(P) \in \mathcal{B}_{11}$ and $\delta(Q) \in \mathcal{B}_{22}$, by Theorem 2.3 $\delta(I) = \delta(P) + \delta(Q)$ commutes with $\mathcal{A}_{11}$, $\mathcal{A}_{12}$ and $\mathcal{A}_{22}$, whence $\delta(I) \in C(\mathcal{T}_{\mathcal{A}}, \mathcal{B})$. Thus $\delta(I) = \lambda I$. By the fact that δ is Jordan derivable at

C, we have $\delta(IC+CI)=\delta(I)C+I\delta(C)+C\delta(I)+\delta(C)I$, which implies $\lambda C=0$. If $C\neq 0$, $\lambda=0$. Hence $\delta(P)=\delta(Q)=0$. If $C=0$, then the fact that $A_{12}A_{11}=0$ holds for every $A_{11}\in\mathcal{A}_{11}$ and $A_{12}\in\mathcal{A}_{12}$ implies that $\delta(A_{11}A_{12})=\delta(A_{12})A_{11}+A_{12}\delta(A_{11})+A_{11}\delta(A_{12})+\delta(A_{11})A_{12}=A_{11}\delta(A_{12})+\delta(A_{11})A_{12}$, which together with faithfulness of $\mathcal{A}_{12}$ leads to $\delta(A_{11}U_{11})=\delta(A_{11})U_{11}+A_{11}\delta(U_{11})$ for every $A_{11},U_{11}\in\mathcal{A}_{11}$. Comparing with Eq. (2.14), we have that $\delta(P)=\delta(Q)=0$.

To see $\delta|_{\mathcal{T}_{\mathcal{A}}}$ is a derivation, it suffices to show that for any $A_{ij},A_{kl}\in\mathcal{T}_{\mathcal{A}}$

$$\delta(A_{ij}A_{kl})=\delta(A_{ij})A_{kl}+A_{ij}\delta(A_{kl}).$$

We will label each case as Case (ij,kl). Since $\delta(A_{11})\in\mathcal{B}_{11}$, $\delta(A_{12})\in\mathcal{B}_{12}$, and $\delta(A_{22})\in\mathcal{B}_{22}$, we only need to check cases for $j=k$. There are only 4 cases.

Case $(11,11)$ follows from Eq. (2.14).

Case $(11,12)$ follows from Eq. (2.11).

Case $(12,22)$ follows from Eq. (2.15).

Case $(22,22)$ follows from Eq. (2.18). □

Corollary 2.6. *Suppose $\mathcal{A}_{12}$ is faithful to $\mathcal{B}$, $\mathcal{A}_{21}=\{0\}$, and $C(\mathcal{A},\mathcal{B})=\mathbb{F}I$. If $\delta\in L(\mathcal{A},\mathcal{B})$ is Jordan derivable at $C\in\mathcal{A}$ then δ is a derivation. In particular, every $C\in\mathcal{A}$ is a Jordan all-derivable point of $L(\mathcal{A},\mathcal{B})$ and every Jordan derivation is a derivation.*

As a consequence of Corollary 2.6, similar to [29, Theorem 4.4] we have

Theorem 2.7. *Let $\mathcal{L}$ be a subspace lattice on a Banach space X and $\mathcal{A}=alg\mathcal{L}$. Suppose there exists a non-trivial idempotent $P\in\mathcal{A}$ such that $ran(P)\in\mathcal{L}$ and $PB(X)(I-P)\subseteq\mathcal{A}$. If $\delta\in L(\mathcal{A},B(X))$ is Jordan derivable at $C\in\mathcal{A}$ then δ is a derivation. In particular, every $C\in\mathcal{A}$ is a Jordan all-derivable point of $L(\mathcal{A},B(X))$.*

Proof. We will apply Corollary 2.6 with $\mathcal{B}=B(X)$. Let $Q=I-P$. The condition $ran(P)\in\mathcal{L}$ implies $\mathcal{A}_{21}=Q\mathcal{A}P=\{0\}$. The condition $PB(X)Q\subseteq\mathcal{A}$ implies $\mathcal{A}_{12}=PB(X)Q$ is faithful. To see $C(\mathcal{A},B(X))=\mathbb{C}I$, take any $B\in C(\mathcal{A},B(X))$, from $BP=PB$ we get $PBQ=0$. From $BQ=QB$, we have $QBP=0$. Thus $B=B_{11}+B_{22}$. For any $x\in ran(P)$ and $f\in X^*$, $x\otimes fQ\in\mathcal{A}_{12}$. It follows from $Bx\otimes fQ=x\otimes fQB$ that $B_{11}x\otimes fQ=x\otimes fQB_{22}$, which leads to $B_{11}x\in\mathbb{C}x$. Since $x\in ran(P)$ is arbitrary, it follows $B_{11}=kP$ for some $k\in\mathbb{C}$. Hence $x\otimes f(kQ-B_{22})=0$, and we have $B_{22}=kQ$ and $B=kI$. Now the conclusion follows from Corollary 2.6. □

As an immediate but noteworthy application of Theorem 2.7, we have the following corollary which generalizes the main result in [33].

Corollary 2.8. *Let $\mathcal{N}$ be a nest on a Banach space X and $\mathcal{A}=alg\mathcal{N}$. Suppose there exists a non-trivial idempotent $P\in\mathcal{A}$ such that $ran(P)\in\mathcal{N}$. If $\delta\in L(\mathcal{A},B(X))$ is Jordan derivable at $C\in\mathcal{A}$ then δ is a derivation. In particular, every $C\in\mathcal{A}$ is a Jordan all-derivable point of $L(\mathcal{A},B(X))$.*

Proof. Let $Q=I-P$. Then $PB(X)Q\subseteq\mathcal{A}$. Now applying Theorem 2.7 completes the proof. □

For an algebra $\mathcal{A}$ and a left $\mathcal{A}$-module $\mathcal{M}$, we call a subset $\mathcal{B}$ of $\mathcal{A}$ *separates* $\mathcal{M}$ if for every $M \in \mathcal{M}$, $\mathcal{B}M = 0$ implies $M = 0$. Let $[\mathcal{A}_{11}, \mathcal{A}_{11}] = \{A_{11}B_{11} - B_{11}A_{11} : \ A_{11}, B_{11} \in \mathcal{A}_{11}\}$.

Theorem 2.9. *Suppose $\mathcal{A}_{12}$ is faithful to $\mathcal{B}$, $C(\mathcal{T}_{\mathcal{A}}, \mathcal{B}) = \mathbb{F}I$, and $[\mathcal{A}_{11}, \mathcal{A}_{11}]$ separates $\mathcal{B}_{12}$. If $\delta \in L(\mathcal{A}, \mathcal{B})$ is Jordan derivable at some $C \in \mathcal{A}_{11} + \mathcal{A}_{12}$ then δ is a derivation. In particular, every $C \in \mathcal{A}_{11} + \mathcal{A}_{12}$ is a Jordan all-derivable point of $L(\mathcal{A}, \mathcal{B})$.*

Proof. Let $C \in \mathcal{A}_{11} + \mathcal{A}_{12}$ and $\delta \in L(\mathcal{A}, \mathcal{B})$ be Jordan derivable at C. Substracting an inner derivation from δ if necessary, we can assume δ satisfies the properties of Theorem 2.3. Let $Q = I - P$ then $QC = 0$. By Theorem 2.3 and Proposition 2.4, $\delta(\mathcal{A}_{11}) \subseteq \mathcal{B}_{11}$, $\delta(\mathcal{A}_{22}) \subseteq \mathcal{B}_{22}$, and $\delta(\mathcal{A}_{12}) \subseteq \mathcal{B}_{12}$; moreover, $\delta(I) = \delta(P) = \delta(Q) = 0$. For any $t \in \mathbb{F}$, $A_{11} \in \mathcal{A}_{11}$ that is invertible in $\mathcal{A}_{11}$ with $A_{11}^{-1} \in \mathcal{A}_{11}$, and $A_{21} \in \mathcal{A}_{21}$, clearly $A_{11}(A_{11}^{-1}C + tA_{21}) = C$. Since δ is Jordan derivable at C, δ differentiates $A_{11} \circ (A_{11}^{-1}C + tA_{21})$. By Proposition 2.2,

$$\delta(A_{11} \circ A_{21}) = \delta(A_{11}) \circ A_{21} + A_{11} \circ \delta(A_{21}) \tag{2.24}$$

Multiplying Q from the right of Eq. (2.24) gives

$$A_{11}\delta(A_{21})Q = \delta(A_{21}A_{11})Q \tag{2.25}$$

For any $U_{11} \in \mathcal{A}_{11}$, by Eq. (2.25) we get

$$A_{11}U_{11}\delta(A_{21})Q = \delta(A_{21}A_{11}U_{11})Q$$

On the other hand, applying Eq. (2.25) twice gives

$$U_{11}A_{11}\delta(A_{21})Q = U_{11}\delta(A_{21}A_{11})Q = \delta(A_{21}A_{11}U_{11})Q$$

It follows $[A_{11}U_{11} - U_{11}A_{11}]\delta(A_{21})Q = 0$. Since $[\mathcal{A}_{11}, \mathcal{A}_{11}]$ separates $\mathcal{B}_{12}$, $P\delta(A_{21})Q = 0$. Multiplying Q from the left of Eq. (2.25) gives $Q\delta(A_{21}A_{11})Q = 0$. In particular, $Q\delta(A_{21})Q = 0$. Multiplying P from both sides of Eq. (2.24) and setting $A_{11} = P$ leads to $P\delta(A_{21})P = 0$. Thus $\delta(A_{21}) \in \mathcal{B}_{21}$.

To see δ is a derivation, it suffices to show that for any $A_{ij} \in \mathcal{A}_{ij}, A_{kl} \in \mathcal{A}_{kl}$

$$\delta(A_{ij}A_{kl}) = \delta(A_{ij})A_{kl} + A_{ij}\delta(A_{kl})$$

We will again label each case as Case (ij, kl). Since $\delta(A_{ij}) \in \mathcal{B}_{ij}$, for all $i, j = 1, 2$, we only need to check cases for $j = k$. There are 8 cases.

Case $(11, 11)$ follows from Eq. (2.14).

Case $(11, 12)$ follows from Eq. (2.11).

Case $(12, 22)$ follows from Eq. (2.15).

Case $(22, 22)$ follows from Eq. (2.18).

Case $(21, 11)$ follows from Eq. (2.24).

It remains to show Cases $(12, 21)$, $(21, 12)$, and $(22, 21)$.

For any $s, t \in \mathbb{F}$, a routine computation shows $(P + sA_{12})[t(A_{21} - sA_{12}A_{21}) - sA_{12} + C + Q] = C$. Since δ is Jordan derivable at C, δ differentiates $(P + sA_{12}) \circ [t(A_{21} - sA_{12}A_{21}) - sA_{12} + C + Q]$.

By Proposition 2.2, δ differentiates $(P + sA_{12}) \circ (A_{21} - sA_{12}A_{21})$. Applying Proposition 2.2 again , we see δ differentiates $P \circ (-A_{12}A_{21}) + A_{12} \circ A_{21}$. Case $(11, 11)$ implies δ differentiates $P \circ (-A_{12}A_{21})$, It follows that δ differentiates $A_{12} \circ A_{21}$, i.e.

$$\delta(A_{12} \circ A_{21}) = \delta(A_{12}) \circ A_{21} + A_{12} \circ \delta(A_{21}) \tag{2.26}$$

Multiplying P from both sides of Eq. (2.26) gives Case $(12, 21)$ and multiplying Q from both sides of Eq. (2.26) gives Case $(21, 12)$.

Applying Case $(21, 12)$, we obtain

$$\delta(A_{22}A_{21}A_{12}) = \delta(A_{22}A_{21})A_{12} + A_{22}A_{21}\delta(A_{12}) \tag{2.27}$$

Using Cases $(22, 22)$ and $(21, 12)$, we have

$$\begin{aligned}\delta(A_{22}A_{21}A_{12}) &= \delta(A_{22})A_{21}A_{12} + A_{22}\delta(A_{21}A_{12}) \\ &= \delta(A_{22})A_{21}A_{12} + A_{22}\delta(A_{21})A_{12} + A_{22}A_{21}\delta(A_{12}) \end{aligned} \tag{2.28}$$

By (2.27) and (2.28), $\delta(A_{22}A_{21})A_{12} = \delta(A_{22})A_{21}A_{12} + A_{22}\delta(A_{21})A_{12}$. Since $\mathcal{A}_{12}$ is faithful, we get $\delta(A_{22}A_{21}) = \delta(A_{22})A_{21} + A_{22}\delta(A_{21})$, completing the proof of Case $(21, 11)$. $\square$

Corollary 2.10. *Suppose H is a Hilbert space and $C \in B(H)$ such that $\ker(C) \neq 0$ or $\ker(C^*) \neq 0$. If $\delta \in L(B(H), B(H))$ is Jordan derivable at C then δ is a derivation. In particular, C is a Jordan all-derivable point of $L(B(H), B(H))$.*

Proof. If $\ker(C^*) \neq 0$, then there exists a proper orthogonal projection $P \in B(H)$ such that $ran(C) \subseteq PH$. Let $Q = I - P$ then $QC = 0$. Take $\mathcal{A} = \mathcal{B} = B(H)$, the one can check that all hypotheses of Theorem 2.9 are satisfied and the conclusions follow.

If $\ker(C) \neq 0$, we can define $\delta^* \in L(B(H), B(H))$ by $\delta^*(A) = (\delta(A^*))^*$ for every $A \in B(H)$. Since δ is Jordan derivable at C, we have δ^* is Jordan derivable at C^*. Now by the argument in the first paragraph we have δ^* is a derivation, and in turn δ is a derivation. This completes the proof. $\square$

3 Jordan higher derivable mappings

In this section, we assume that $\mathcal{A}$ is an algebra over a field $\mathbb{F}$ of characteristic zero. Before stating our main results in this section, we first need a proposition that characterizes Jordan higher derivations in terms of Jordan derivations. Since the proof is similar to the proof of [26, Theorem 2.5], we omit it here.

Proposition 3.1. *Let $\mathcal{A}$ be an algebra, $\{d_i\}_{i\in\mathbb{N}}$ be a sequence of mappings on $\mathcal{A}$ with $d_0 = I_{\mathcal{A}}$ and $\{\delta_i\}_{i\in\mathbb{N}}$ be the a sequences of (Jordan) derivations on $\mathcal{A}$ with $\delta_0 = 0$. If the following recursive relation holds:*

$$nd_n = \sum_{k=0}^{n-1} \delta_{k+1}d_{n-1-k}$$

for $n \geq 1$, then $\{d_i\}_{i\in\mathbb{N}}$ is a (Jordan) higher derivation.

Let $\mathcal{R}(\mathcal{A})$ be a relation on $\mathcal{A}$, i.e. $\mathcal{R}(\mathcal{A})$ is a nonempty subset of $\mathcal{A} \times \mathcal{A}$. We say $\delta \in L(\mathcal{A},\mathcal{M})$ is *derivable on* $\mathcal{R}(\mathcal{A})$ if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $(A,B) \in \mathcal{R}(\mathcal{A})$. We say $\delta \in L(\mathcal{A},\mathcal{M})$ is *Jordan derivable on* $\mathcal{R}(\mathcal{A})$ if $\delta(AB+BA) = \delta(A)B+A\delta(B)+\delta(B)A+B\delta(A)$ for all $(A,B) \in \mathcal{R}(\mathcal{A})$. A sequence of mappings $\{d_i\}_{i\in\mathbb{N}} \in L(\mathcal{A})$ with $d_0 = I_{\mathcal{A}}$ is called *higher derivable on* $\mathcal{R}(\mathcal{A})$ if $d_n(AB) = \sum_{i+j=n} d_i(A)d_j(B)$ for all $(A,B) \in \mathcal{R}(\mathcal{A})$. A sequence of mappings $\{d_i\}_{i\in\mathbb{N}} \in L(\mathcal{A})$ with $d_0 = I_{\mathcal{A}}$ is called *Jordan higher derivable on* $\mathcal{R}(\mathcal{A})$ if $d_n(AB+BA) = \sum_{i+j=n}(d_i(A)d_j(B)+d_i(B)d_j(A))$ for all $(A,B) \in \mathcal{R}(\mathcal{A})$. We say $\mathcal{R}(\mathcal{A})$ is *(Jordan) derivational* for $L(\mathcal{A},\mathcal{M})$ if every (Jordan) derivable mapping on $\mathcal{R}(\mathcal{A})$ is a (Jordan) derivation. We say $\mathcal{R}(\mathcal{A})$ is *(Jordan) higher derivational* for $L(\mathcal{A})$ if every (Jordan) higher derivable mapping on $\mathcal{R}(\mathcal{A})$ is a (Jordan) higher derivation.

Remark 3.2. The above definitions allow us to unify some of the notions in the literature. For example, in literature, there are two definitions of Jordan derivable mappings, one is what we use in this paper (see for example [6, 33] and references therein), and the other (see for example [2, 13, 31]) is what we call S-Jordan derivable mappings (S stands for standard). A mapping $\delta \in L(\mathcal{A},\mathcal{M})$ is called a *S-Jordan derivable mapping at* $C \in \mathcal{A}$ if $\delta(AB+BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ for all $A, B \in \mathcal{A}$ with $AB+BA = C$. An element $C \in \mathcal{A}$ is called a *S-Jordan all-derivable point* if every S-Jordan derivable mapping at C is a Jordan derivation. A sequence of mappings $\{d_i\}_{i\in\mathbb{N}} \in L(\mathcal{A})$ with $d_0 = I_{\mathcal{A}}$ is called *S-Jordan higher derivable at* $C \in \mathcal{A}$ if $d_n(AB+BA) = \sum_{i+j=n}(d_i(A)d_j(B) + d_i(B)d_j(A))$ for all $A, B \in \mathcal{A}$ with $AB+BA = C$. An element $C \in \mathcal{A}$ is called a *S-Jordan higher all-derivable point* if every sequence of S-Jordan higher derivable mappings at C is a Jordan higher derivation. The above two notions of Jordan derivable mappings at C are special case of Jordan derivable mappings on $\mathcal{R}(\mathcal{A})$, where $\mathcal{R}(\mathcal{A}) = \{(A,B) \in \mathcal{A}\times\mathcal{A} : AB = C\}$ and $\mathcal{R}(\mathcal{A}) = \{(A,B) \in \mathcal{A}\times\mathcal{A} : AB+BA = C\}$, respectively.

Theorem 3.3. *If $\mathcal{A}$ is an algebra such that $\mathcal{R}(\mathcal{A})$ is (Jordan) derivational for $L(\mathcal{A})$, then $\mathcal{R}(\mathcal{A})$ is (Jordan) higher derivational.*

Proof. First, suppose $\mathcal{R}(\mathcal{A})$ is Jordan derivational and $\{d_i\}_{i\in\mathbb{N}}$ is a sequence of mappings in $L(\mathcal{A})$ Jordan higher derivable on $\mathcal{R}(\mathcal{A})$. Let $\delta_1 = d_1$ and $\delta_n = nd_n - \sum_{k=0}^{n-2}\delta_{k+1}d_{n-1-k}$ for every $n(\geq 2) \in \mathbb{N}$. We will show $\{\delta_i\}_{i\in\mathbb{N}}$ is a sequence of Jordan derivations, and in turn $\{d_i\}_{i\in\mathbb{N}}$ is a Jordan higher derivation by Proposition 3.1. We prove by induction.

When $n = 1$, since $\mathcal{R}(\mathcal{A})$ is Jordan derivational, we have that δ_1 is a Jordan derivation.

Now suppose δ_k is defined as above and is a Jordan derivation for $k \leq n$. For $(A,B) \in \mathcal{R}(\mathcal{A})$, we have

$$\begin{aligned}\delta_{n+1}(A\circ B) &= (n+1)d_{n+1}(A\circ B) - \sum_{k=0}^{n-1}\delta_{k+1}d_{n-k}(A\circ B)\\ &= (n+1)\sum_{k=0}^{n+1}\{d_k(A)\circ d_{n+1-k}(B)\} - \sum_{k=0}^{n-1}\delta_{k+1}\sum_{l=0}^{n-k}\{d_l(A)\circ d_{n-k-l}(B)\}.\end{aligned}$$

By induction we have

$$\delta_{n+1}(A\circ B)=\sum_{k=0}^{n+1}kd_k(A)\circ d_{n+1-k}(B)+\sum_{k=0}^{n+1}d_k(A)\circ(n+1-k)d_{n+1-k}(B)$$
$$-\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\{\delta_{k+1}(d_l(A))\circ d_{n-k-l}(B)+d_l(A)\circ\delta_{k+1}(d_{n-k-l}(B))\}.$$

Set

$$K_1=\sum_{k=0}^{n+1}kd_k(A)\circ d_{n+1-k}(B)-\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\delta_{k+1}(d_l(A))\circ d_{n-k-l}(B),$$
$$K_2=\sum_{k=0}^{n+1}d_k(A)\circ(n+1-k)d_{n+1-k}(B)-\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}d_l(A)\circ\delta_{k+1}(d_{n-k-l}(B)).$$

Then $\delta_{n+1}(A\circ B)=K_1+K_2$. Let us compute K_1 and K_2. If we put $r=k+l$ in the summation $\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}$, then we may write it as $\sum_{r=0}^{n}\sum_{0\le k\le r,k\neq n}$. Hence

$$K_1=\sum_{k=0}^{n+1}kd_k(A)\circ d_{n+1-k}(B)-\sum_{r=0}^{n}\sum_{0\le k\le r,k\neq n}\delta_{k+1}(d_{r-k}(A))\circ d_{n-r}(B).$$

Putting $r+1$ instead of k in the first summation, we have

$$\begin{aligned}K_1&+\sum_{k=0}^{n-1}\delta_{k+1}(d_{n-k}(A))\circ B\\&=\sum_{r=0}^{n}(r+1)d_{r+1}(A)\circ d_{n-r}(B)-\sum_{r=0}^{n-1}\sum_{k=0}^{r}\delta_{k+1}(d_{r-k}(A))\circ d_{n-r}(B)\\&=\sum_{r=0}^{n-1}\{(r+1)d_{r+1}(A)-\sum_{k=0}^{r}\delta_{k+1}(d_{r-k}(A))\}\circ d_{n-r}(B)+(n+1)d_{n+1}(A)\circ B.\end{aligned}$$

By our assumption $(r+1)d_{r+1}(A)=\sum_{k=0}^{r}\delta_{k+1}(d_{r-k}(A))$ for $r=0,\dots,n-1$, we obtain

$$K_1=(n+1)d_{n+1}(A)\circ B-\sum_{k=0}^{n-1}\delta_{k+1}(d_{n-k}(A))\circ B=\delta_{n+1}(A)\circ B.$$

Similary, we may deduce that

$$K_2=(n+1)A\circ d_{n+1}(B)-\sum_{k=0}^{n-1}A\circ\delta_{k+1}(d_{n-k}(B))=A\circ\delta_{n+1}(B).$$

Therefore, δ_{n+1} is Jordan derivable on $\mathcal{R}(\mathcal{A})$. Since $\mathcal{R}(\mathcal{A})$ is Jordan derivational, we have that δ_{n+1} is a Jordan derivation.

Similarly, we can prove the case when $\mathcal{R}(\mathcal{A})$ is assumed to be derivational by changing " $\circ$" to the normal multiplication of $\mathcal{A}$. $\square$

Recall that a mapping $\delta \in L(\mathcal{A}, \mathcal{M})$ is called *derivable at* $C \in \mathcal{A}$ if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$ with $AB = C$. An element $C \in \mathcal{A}$ is called an *all-derivable point* if every derivable mapping at C is a derivation. A sequence of mappings $\{d_i\}_{i\in\mathbb{N}} \in L(\mathcal{A})$ with $d_0 = I_{\mathcal{A}}$ is called *higher derivable at* $C \in \mathcal{A}$ if $d_n(AB) = \sum_{i+j=n} d_i(A)d_j(B)$ for all $A, B \in \mathcal{A}$ with $AB = C$. An element $C \in \mathcal{A}$ is called a *higher all-derivable point* if every sequence of higher derivable mappings at C is a higher derivation.

Remark 3.4. Several authors (see for example [1, 11, 12, 18, 19, 24, 30, 36, 37, 38, 39]) investigate derivable mappings at 0, invertible element, left (right) separating point, non-trivial idempotent, and the unit I on certain algebras. By Theorem 3.3, we can generalize these results to the higher derivation case. Many authors also study (S-)Jordan derivable mappings (see for example [2, 6, 13, 31, 33]) at these points. Theorems 3.3 also allow us to generalize these results to the (S-)Jordan higher derivation case.

Combining Theorem 3.3 with Corollary 2.6, we have

Corollary 3.5. *Suppose $\mathcal{A}$, $\mathcal{B}$ are as in Corollary 2.6 with $\mathcal{B} = \mathcal{A}$. Then every $C \in \mathcal{A}$ is a Jordan higher all-derivable point.*

Combining Theorem 3.3 with Theorem 2.7, we have

Corollary 3.6. *Let $\mathcal{L}$ be a subspace lattice on a Banach space X and $\mathcal{A} = alg\mathcal{L}$. If there exists a non-trivial idempotent $P \in \mathcal{A}$ such that $ran(P) \in \mathcal{L}$ and $PB(X)(I - P) \subseteq \mathcal{A}$, then every $C \in \mathcal{A}$ is a Jordan higher all-derivable point.*

Combining Theorem 3.3 with [29, Theorem 3.3], we have

Corollary 3.7. *Suppose $\mathcal{A}$, $\mathcal{B}$ are as in Corollary 2.6 with $\mathcal{B} = \mathcal{A}$. Then every $0 \neq C \in \mathcal{A}$ is a higher all-derivable point .*

We say that W in an algebra $\mathcal{A}$ is a left (or right) separating point of $\mathcal{A}$ if $WA = 0$ (or $AW = 0$) for $A \in \mathcal{A}$ implies $A = 0$. In [20, Remark 1], the authors point out that if every Jordan derivation on a unital Banach algebra $\mathcal{A}$ is a derivation, then every linear mapping on $\mathcal{A}$ which is derivable at an arbitrary left or right separating point of $\mathcal{A}$ is a derivation. Together with Theorem 3.3, we may generalize this result to the higher derivation case.

Theorem 3.8. *Let $\mathcal{A}$ be a unital Banach algebra such that every Jordan derivation on $\mathcal{A}$ is a derivation. Suppose that W in $\mathcal{A}$ is a left or right separating point. If $D = (d_i)_{i\in\mathbb{N}}$ is a family of linear mappings higher derivable at W, then $D = (d_i)_{i\in\mathbb{N}}$ is a higher derivation.*

For any non-zero vectors $x \in X$ and $f \in X^*$, the rank one operator $x \otimes f$ is defined by $x \otimes f(y) = f(y)x$ for $y \in X$.

Lemma 3.9. *If $\mathcal{A}$ is a norm-closed subalgebra of $B(X)$ such that $\vee\{x : x \otimes f \in \mathcal{A}\} = X$ and $\wedge\{\ker(f) : x \otimes f \in \mathcal{A}\} = (0)$, then every derivation δ from $\mathcal{A}$ into $B(X)$ is bounded.*

Proof. By the closed graph theorem, it is sufficient to show if $A_n \to A$ and $\delta(A_n) \to B$, as $n \to \infty$, then $\delta(A) = B$.

For any $x\otimes f,\ y\otimes g\in\mathcal{A}$, since

$$\begin{aligned}\delta(x\otimes fA_ny\otimes g)&=f(A_ny)\delta(x\otimes g)\\&=x\otimes f\delta(A_ny\otimes g)+\delta(x\otimes f)(A_ny\otimes g)\\&=x\otimes f(\delta(A_n)y\otimes g+A_n\delta(y\otimes g))+\delta(x\otimes f)(A_ny\otimes g),\end{aligned}$$

we have

$$(x\otimes f)\delta(A_n)(y\otimes g)=f(A_ny)\delta(x\otimes g)-(x\otimes f)A_n\delta(y\otimes g)-\delta(x\otimes f)(A_ny\otimes g). \tag{3.1}$$

Taking limit in (3.1) yields

$$(x\otimes f)B(y\otimes g)=(x\otimes f)\delta(A)(y\otimes g).$$

Hence $f(By)=f(\delta(A))$. Thus $\delta(A)=B$. $\square$

By [27], if $\{d_i\}_{i\in\mathbb{N}}$ is a Jordan higher derivation on an algebra $\mathcal{A}$, then there is a sequence $\{\delta_i\}_{i\in\mathbb{N}}$ of Jordan derivations on $\mathcal{A}$ such that

$$d_n=\sum_{i=1}^{n}\left(\sum_{\sum_{j=1}^{i}r_j=n}\left(\prod_{j=1}^{i}\frac{1}{r_j+\cdots r_i}\right)\delta_{r_1}\ldots\delta_{r_i}\right),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i}r_j=n$. This together with Lemma 3.9 leads to the following Theorem.

Theorem 3.10. *If $\mathcal{A}$ is a norm-closed subalgebra of $B(X)$ such that $\vee\{x : x\otimes f\in\mathcal{A}\}=X$ and $\wedge\{\ker(f) : x\otimes f\in\mathcal{A}\}=(0)$, then every Jordan higher derivation on alg$\mathcal{L}$ is bounded.*

Proof. Since every Jordan derivation on $\mathcal{A}$ is a derivation by [18, Theorem 4.1]. $\square$

For a subspace lattice $\mathcal{L}$ of a Banach space X and for $E\in\mathcal{L}$, define

$$E_-=\vee\{F\in\mathcal{L} : F\not\supseteq E\}.$$

Put

$$\mathcal{J}(\mathcal{L})=\{K\in\mathcal{L} : K\neq(0)\text{ and }K_-\neq X\}.$$

Remark 3.11. It is well known (see [21]) that $x\otimes f\in\text{alg}\mathcal{L}$ if and only if there exists some $K\in\mathcal{J}(\mathcal{L})$ such that $x\in K$ and $f\in K_-^{\perp}$. It follows that if a subspace lattice $\mathcal{L}$ satisfies $\vee\{K : K\in\mathcal{J}(\mathcal{L})\}=H$ and $\wedge\{K_- : K\in\mathcal{J}(\mathcal{L})\}=(0)$, then *alg*$\mathcal{L}$ satisfies the hypothesis of Theorem 3.10. Such subspace lattices include completely distributive subspace lattices, $\mathcal{J}$-subspace lattices, and subspace lattices with $H_-\neq H$ and $(0)_+\neq(0)$. Recall that (see [22]), a subspace lattice $\mathcal{L}$ is called *completely distributive* if $L=\vee\{E\in\mathcal{L} : E_-\not\supseteq L\}$ and $L=\wedge\{E_- : E\in\mathcal{L}\text{ and }E\not\leq L\}$ for all $L\in\mathcal{L}$. It follows that completely distributive subspace lattices satisfy the conditions $\vee\{K : K\in\mathcal{J}(\mathcal{L})\}=H$ and $\wedge\{K_- : K\in\mathcal{J}(\mathcal{L})\}=(0)$. A subspace lattice $\mathcal{L}$ is called a *$\mathcal{J}$-subspace lattice* on H if $\vee\{K : K\in\mathcal{J}(\mathcal{L})\}=H$, $\wedge\{K_- : K\in\mathcal{J}(\mathcal{L})\}=(0)$, $K\vee K_-=H$ and $K\wedge K_-=(0)$ for any $K\in\mathcal{J}(\mathcal{L})$.

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